MTH 132 Section 103 Exam 1

Name: Solutions

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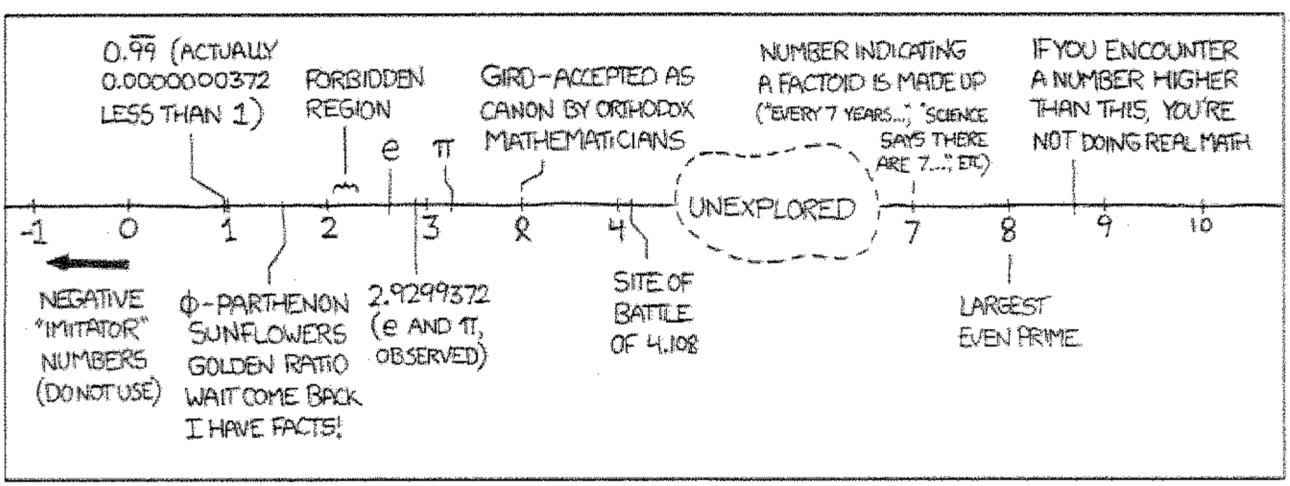
Do not start this exam until instructed; you will have 50 minutes to finish the exam. Show all your work to receive credit. No notes, books, calculators, phones or electronic devices are allowed on this exam.

There are 7 problems on this exam on 5 pages, in addition to this cover page.

Good luck!



From *xkcd*.

1. (15 points) Consider the function $f(x) = x^2 + x$.

(a) Find the average rate of change of f on the interval $[1, 1+h]$.

$$\text{AROC} = \frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{[(1+h)^2 + (1+h)] - [1^2 + 1]}{h}$$

$$= \frac{\cancel{1} + 2h + h^2 + \cancel{1} + h - (\cancel{1} + \cancel{1})}{h}$$

$$= \frac{3h + h^2}{h}$$

$$= \boxed{3 + h}$$

(b) Find the limit of the average rate of change as h approaches 0.

$$\lim_{h \to 0} \text{AROC} = \lim_{h \to 0} 3 + h = \boxed{3.}$$

(c) Find an expression for the tangent line for f at the point $(1, 2)$.

Slope = 3 from above

$\longrightarrow y = 3x + b.$

Use the point $(1, 2)$:

$2 = 3 \cdot 1 + b$

$\rightarrow$

$b = -1$ $\boxed{y = 3x - 1}$

2. (24 points) Evaluate the following limits, justifying your steps using known limit laws or results. If the limit does not exist, state why.

(a) $\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3}$

Multiply by the conjugate:

$$\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3} \cdot \frac{\sqrt{x+6}+3}{\sqrt{x+6}+3} = \lim_{x\to 3} \frac{\overbrace{(x+6)-3}^{x-3}}{(x-3)(\sqrt{x+6}+3)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+6}+3} = \frac{1}{\sqrt{3+6}+3} = \boxed{\frac{1}{12}}$$

(b) $\lim_{t\to 0} \frac{\sin(4t^2)}{2t}$

Multiply top & bottom by $2t$:

$$\lim_{t\to 0} \frac{\sin(4t^2)}{2t} = \lim_{t\to 0} \left(\frac{\sin(4t^2)}{4t^2} \cdot 2t\right)$$

$$= \left(\lim_{t\to 0} \frac{\sin(4t^2)}{4t^2}\right)\left(\lim_{t\to 0} 2t\right) = 1 \cdot 0 = \boxed{0}$$

(c) $\lim_{w\to 0} \frac{w}{|w|}$

If $w > 0$, $\frac{w}{|w|} = 1 \Rightarrow \lim_{w\to 0^+} \frac{w}{|w|} = 1$

If $w < 0$, $\frac{w}{|w|} = -1 \Rightarrow \lim_{w\to 0^-} \frac{w}{|w|} = -1$

Not Equal!

So $\boxed{\lim_{w\to 0} \frac{w}{|w|} \text{ doesn't exist.}}$

(d) Given that f is a function satisfying $-x^2 \le f(x) \le |\sin x|$ for all x, can you say whether $\lim_{x\to 0} f(x)$ exists? If so, what is the limit?

Since $\lim_{x\to 0} (-x^2) = 0$ and $\lim_{x\to 0} |\sin x| = 0$, the Sandwich Theorem implies that $\boxed{\lim_{x\to 0} f(x) = 0}$

3. (10 points) Consider the following limit statement:

$$\lim_{x \to 6} (3x - 12) = 6$$

Given $\epsilon > 0$, find the largest value of $\delta > 0$ so that the formal definition of the limit holds.

Set $g(x) = 3x - 12$. We want

$0 < |x - 6| < \delta \Rightarrow |g(x) - 6| < \epsilon$.

$0 < |x - 6| < \delta \Rightarrow |(3x - 12) - 6| < \epsilon$

$0 < |x - 6| < \delta \Rightarrow |3x - 18| < \epsilon$

$0 < |x - 6| < \delta \Rightarrow 3|x - 6| < \epsilon$

$0 < |x - 6| < \delta \Rightarrow |x - 6| < \frac{\epsilon}{3}$

So we set $\boxed{\delta = \epsilon/3}$.

4. (10 points) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{if } x \neq 5 \\ 11, & \text{if } x = 5 \end{cases}$$

Is f continuous at $x = 5$? Why or why not?

Note $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{x - 5}$

$= \lim_{x \to 5} x + 5 = 10.$

~~So~~ $\cancel{\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = 10}$

So $\lim_{x \to 5} f(x) = 10 \neq 11 = f(5)$

so $\boxed{f \text{ is not continuous at } 5.}$

5. (15 points) Consider the function

$$f(x) = \frac{x^3}{|x|^3 - 27}$$

(a) Evaluate

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

Write $f(x) = \dfrac{x^3}{x^3 - 27}$ if $x > 0$

$= \dfrac{x^3}{-x^3 - 27}$ if $x < 0$.

So if $x > 0$, $f(x) = \dfrac{1}{1 - 27/x^3} \longrightarrow 1$ as $x \to \infty$; likewise, $f(x) \longrightarrow -1$ if $x \to -\infty$.

(b) Find all horizontal and vertical asymptotes of f.

Horizontal: $\boxed{y = \pm 1}$ from above.

Vertical: Denominator $= 0 \Rightarrow |x|^3 = 27$

$\Rightarrow |x| = 3 \Rightarrow \boxed{x = \pm 3}$

(c) Where is the function f continuous? Justify your answer.

It's a quotient of continuous functions, so it's continuous except when the denominator is zero, at $x = \pm 3$. So f is continuous on $\boxed{(-\infty, -3) \cup (-3, 3) \cup (3, \infty).}$

6. (6 points) Prove that the equation $\pi \cos x = x$ has a solution in the interval $[0, \pi]$. *Hint: What can you say about the function* $g(x) = \pi \cos x - x$*?*

g is continuous,

$g(0) = \pi > 0$

$g(\pi) = \pi \cos \pi - \pi$

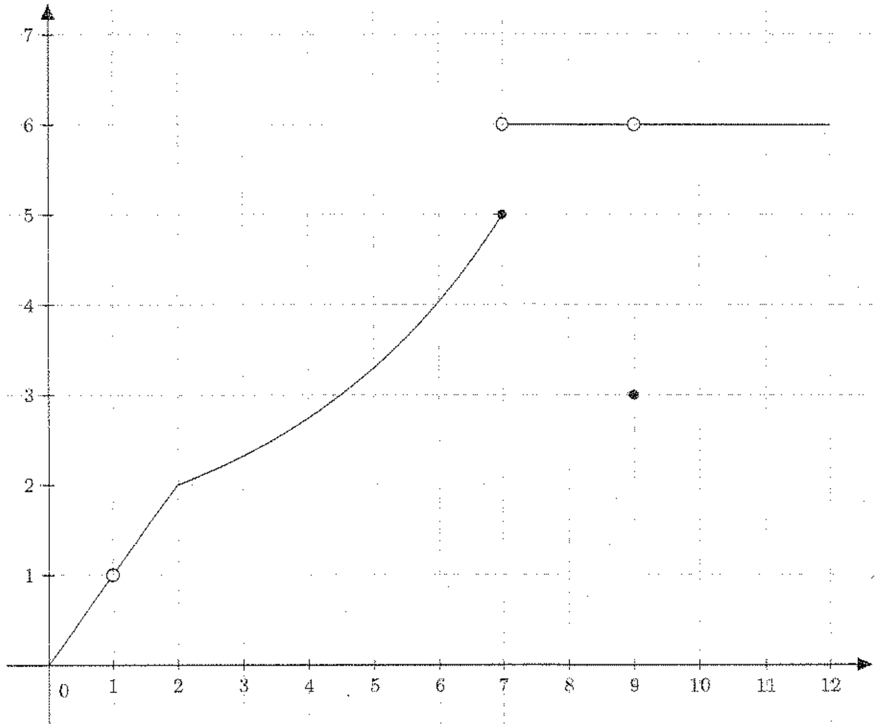
$= -\pi - \pi$

$= -2\pi < 0$

So there exists α, $0 \le \alpha \le \pi$ with $g(\alpha) = 0$.

Then α is a solution.

7. (20 points) Consider the following graph of a function f, defined on $[0, 1) \cup (1, 12]$, and answer the following questions about f. **You do not need to show any work for this problem.**



(a) What is $\lim_{x \to 1} f(x)$?

$\boxed{1}$

(b) What is $\lim_{x \to 9} f(x)$?

$\boxed{6}$

(c) Find the one-sided limits $\lim_{x \to 7^+} f(x)$ and $\lim_{x \to 7^-} f(x)$. Does $\lim_{x \to 7} f(x)$ exist?

$\lim_{x \to 7^+} f(x) = 6 \neq \lim_{x \to 7^-} f(x) = 5.$ So $\boxed{\text{no}}$.

(d) For what values of x, between 0 and 12, is f *not* continuous?

$\boxed{x = 1, 7, 9.}$